

Three-dimensional Gaussian fluctuations of non-commutative random surfaces along time-like paths

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Abstract

We construct a continuous-time non-commutative random walk on $U(\mathfrak{gl}_N)$ with dilation maps $U(\mathfrak{gl}_N) \rightarrow L^2(U(N))^{\otimes \infty}$. This is an analog of a continuous-time non-commutative random walk on the group von Neumann algebra $vN(U(N))$ constructed in [15], and is a variant of discrete-time non-commutative random walks on $U(\mathfrak{gl}_N)$ [2, 9].

It is also shown that when restricting to the Gelfand–Tsetlin subalgebra of $U(\mathfrak{gl}_N)$, the non-commutative random walk matches a (2+1)-dimensional random surface model introduced in [7]. As an application, it is then proved that the moments converge to an explicit Gaussian field along time-like paths. Combining with [7] which showed convergence to the Gaussian free field along space-like paths, this computes the entire three-dimensional Gaussian field. In particular, it matches a Gaussian field from eigenvalues of random matrices [5].

1 Introduction

Let us review some results in the mathematical and physics literature in order to motivate the problem.

The *Anisotropic Kardar–Parisi–Zhang* (AKPZ) equation, which was introduced in [20] and is a variant of the KPZ equation first considered in [12], describes a universal class of random surface growth models. Letting $h(t)$ denote the height of the surface at time t , the equation in two dimensions is

$$\partial_t h = \nu_x \partial_x^2 h + \nu_y \partial_y^2 h + \frac{1}{2} \lambda_x (\partial_x h)^2 + \frac{1}{2} \lambda_y (\partial_y h)^2 + \eta,$$

where η is space-time white noise and λ_x, λ_y have different signs. (When λ_x and λ_y have the same sign, the equation is just the usual KPZ equation in two dimensions). Using non-rigorous methods, it was predicted (e.g. [13]) that the stationary distribution for the AKPZ dynamics would be the Gaussian free field (see [17] for a mathematical approach to the Gaussian free field). The question about the full three-dimensional process across different time variables remained open.

However, the equation is mathematically ill-defined, due to the non-linear term. One mathematical approach is to consider exactly solvable models in the AKPZ universality class. There have been two models considered, an interacting particle system and the eigenvalue process of a random matrix. Both will be described now.

The interacting particle system, studied in [7], lives on the lattice $\mathbb{Z} \times \mathbb{Z}_+$. It was shown that along *space-like paths*, the particle system is a determinantal point process. (See Theorem 5.1 for the definition of space-like paths). By computing the correlation kernel and taking asymptotics, it was shown that the fluctuations of the height function of the particle system indeed converge to the Gaussian free field. But due to the space-like path restriction, the problem of computing the limiting three-dimensional field remained unsolved.

The random matrix model looks at the eigenvalues of minors of a large random matrix whose entries are evolving as Ornstein–Uhlenbeck processes. By a combinatorial argument, [5] was able to compute the limiting three-dimensional Gaussian field, which has the Gaussian free field as a stationary distribution. The asymptotics at the edge were also computed in [18]. However, one drawback is that the eigenvalues are not Markovian, as shown in [1].

With these two models in mind, it is natural to want to consider an exactly solvable model that “combines” both models, and which is both Markovian and allows for the limiting three-dimensional field to be computed. This paper will construct such a model.

Let us outline the body of the paper. First, the model will be constructed as a continuous-time non-commutative random walk, which is a non-commutative version of the usual random walk in classical probability. The “state space” is $U(\mathfrak{gl}_N)$, the universal enveloping algebra of the Lie algebra \mathfrak{gl}_N of $N \times N$ matrices. The dilation maps are algebra homomorphisms $j_n : U(\mathfrak{gl}_N) \rightarrow (M^{\otimes \infty}, \omega)$, where M is a von Neumann sub-algebra of the $U(\mathfrak{gl}_N)$ -module $L^2(U(N))$ and ω is a state on $M^{\otimes \infty}$. These j_n are a non-commutative analog of the usual definition of a stochastic process as a family of maps X_n from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to a state space S . It is proved below (Theorem 3.1) that there is a semi-group of non-commutative Markov operator $\{P_t\}_{t \geq 0}$ on $U(\mathfrak{gl}_N)$ which is consistent with j_n .

This model is analogous to a previously constructed non-commutative random walk on the group von Neumann algebra $vN(U(N))$ with dilation maps $vN(U(N)) \rightarrow vN(U(N))^{\otimes \infty}$ [15]. Additionally, it preserves the states from [6]. All of these construction involve a continuous family of characters of the infinite-dimensional unitary group $U(\infty)$. There have also been previous non-commutative random walks using the basic representation of $U(N)$ as input [2, 9].

It also turns out that P_t preserves $Z := Z(U(\mathfrak{gl}_N))$, the centre of $U(\mathfrak{gl}_N)$. This means that $P_t|_Z$ is a Markov operator in the usual (classical) sense. This Markov operator has a natural description: By using the Harish–Chandra isomorphism which identifies Z with the ring of shifted symmetric polynomials in N variables, P_t can be identified with the Markov operator Q_t of an interacting system of N particles on \mathbb{Z} . This

is shown in Proposition 4.4 below. This interacting system is known as the *Charlier Process*, see [14]. In fact, the projection of the interacting particle system from [7] onto $\mathbb{Z} \times \{N\}$ is exactly Q_t . When restricting our non-commutative random walk to the Gelfand–Tsetlin subalgebra, which is the subalgebra of $U(\mathfrak{gl}_N)$ generated by the centres $Z(U(\mathfrak{gl}_k))$, $1 \leq k \leq N$, it also matches the two-dimensional particle system along space-like paths; see Theorem 4.5 for the precise statement. It is worth mentioning that the matching most likely does not hold along time-like paths.

We then take asymptotics of certain elements of the Gelfand–Tsetlin subalgebra and prove convergence to jointly Gaussian random variables. These elements correspond to moments of the random surface. Here, there is no requirement that the paths be space-like, allowing for convergence to Gaussians along time-like paths as well. The explicit covariance formula is given in Theorem 5.2.

At first glance, it appears to be slightly different from the covariance formula for eigenvalues of random matrices. However, the process here corresponds to Brownian motion (see e.g. [4, 9]). Indeed, after applying the usual rescaling from Brownian motion to Ornstein–Uhlenbeck, the covariance from [5] is recovered.

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2 Preliminaries

Let us review some background about representation theory and non-commutative random walks. See [3] for an introduction to non-commutative random walks.

2.1 Representation Theory

The universal enveloping algebra $U(\mathfrak{gl}_N)$ is the unital algebra over \mathbb{C} generated by $\{E_{ij}, 1 \leq i, j \leq N\}$ with relations $E_{ij}E_{kl} - E_{kl}E_{ij} = \delta_{jk}E_{il} - \delta_{il}E_{kj}$. It carries a natural $*$ -operation induced from complex conjugation on \mathbb{C} . The coproduct $\Delta : U(\mathfrak{gl}_N) \rightarrow U(\mathfrak{gl}_N) \otimes U(\mathfrak{gl}_N)$ is the algebra morphism sending E_{ij} to $E_{ij} \otimes 1 + 1 \otimes E_{ij}$. There is a natural one-to-one correspondence between finite-dimensional $U(\mathfrak{gl}_N)$ -modules, finite-dimensional Lie algebra representations of \mathfrak{gl}_N , and finite-dimensional representations of the Lie group $G := U(N)$.

Let $L^2(G)$ be the Hilbert space of square-integrable complex-valued functions on G . Recall that by the Peter–Weyl theorem, this Hilbert space has an orthogonal basis given by the matrix coefficients of all irreducible representations of G , i.e.

$$\{g \mapsto \eta(\pi_\lambda(g)\xi)\},$$

where π_λ runs over all irreducible representations of G , $\{\xi\}$ runs over a basis for V_λ and $\{\eta\}$ runs over a basis for V_λ^* . Denote this basis by $\{f_{\xi\eta}\}$. Then there is a non-degenerate pairing $\langle \cdot, \cdot \rangle$ between $U(\mathfrak{gl}_N)$ and $L^2(G)$

given by

$$\langle X, f_{\xi\eta} \rangle = \eta(X\xi).$$

This can be heuristically understood as $\langle X, f \rangle = f(X)$, since $f_{\xi\eta}(g) = \eta(g\xi)$. This pairing defines an injection $U(\mathfrak{gl}_N) \hookrightarrow L^2(G)^*$. Let us review the algebra structure of $L^2(G)^*$.

There is a co-algebra structure on $L^2(G)$ given by the co-product $\Delta : L^2(G) \rightarrow L^2(G) \otimes L^2(G) \cong L^2(G \times G)$ defined by $\Delta(f)(x, y) = f(xy)$. The multiplication μ on $L^2(G)^*$ is the composition

$$L^2(G)^* \otimes L^2(G)^* \xrightarrow{\rho} (L^2(G) \otimes L^2(G))^* \xrightarrow{\Delta^*} L^2(G)^*,$$

where $\rho(\phi \otimes \psi)(f \otimes h) = \phi(f)\psi(h)$. Use Sweedler's notation to write

$$\Delta(f) = \sum_{(f)} f_{(1)} \otimes f_{(2)}.$$

Evaluating both sides at $(x, y) \in G \times G$ shows

$$f(xy) = \sum_{(f)} f_{(1)}(x)f_{(2)}(y) \text{ for all } x, y \in G.$$

Then

$$\mu(\phi \otimes \psi)(f) = \Delta^* \rho(\phi \otimes \psi)(f) = \rho(\phi \otimes \psi)(\Delta f) = \sum_{(f)} \phi(f_{(1)})\psi(f_{(2)}).$$

In particular, if $\phi_x \in L^2(G)^*$ denotes evaluation at x , i.e. $\phi_x(f) = f(x)$, then

$$(\phi_x \phi_y)(f) = \sum_{(f)} \phi_x(f_{(1)})\phi_y(f_{(2)}) = \sum_{(f)} f_{(1)}(x)f_{(2)}(y) = f(xy).$$

So $\phi_x \phi_y = \phi_{xy}$. We also write $\phi_X(\cdot)$ for $\langle X, \cdot \rangle$.

With the pairing between $U(\mathfrak{gl}_N)$ and $L^2(G)$ above, define the action π of $U(\mathfrak{gl}_N)$ on $L^2(G)$ by

$$\pi(a) : f \mapsto \langle \text{id} \otimes a, \Delta f \rangle.$$

The symbol π will sometimes be repressed, in the sense that af means $\pi(a)f$. Observe that π preserves each summand in the Peter-Weyl decomposition $L^2(G) = \bigoplus_{\lambda} V_{\lambda}^{(1)} \oplus \dots \oplus V_{\lambda}^{(\dim(\lambda))}$. To see this, suppose we are given some matrix coefficient in an irreducible representation V_{λ} , that is, an $f \in L^2(G)$ of the form

$$f(g) = \langle gv, w \rangle \text{ for fixed } v, w \in V_{\lambda}.$$

Then by the definition of the co-product

$$\sum_{(f)} f_{(1)}(g_1)f_{(2)}(g_2) = \langle g_1 g_2 v, w \rangle.$$

Since $\langle X, f_{(2)} \rangle = f_{(2)}(X)$, we see that

$$(\pi(X)f)(g) = \langle g \cdot Xv, w \rangle. \quad (1)$$

Thus, $\pi(X)$ is of the form $\langle gv', w \rangle$ for $v', w \in V_\lambda$, so the summand is preserved. Letting be the von Neumann algebra consisting of the elements of $\text{Hom}_\mathbb{C}(L^2(G), L^2(G))$ which preserve each summand in the Peter–Weyl decomposition, we have that π sends $U(\mathfrak{gl}_N)$ to M . From the definition of the co-product in $U(\mathfrak{gl}_N)$, the n -th tensor power $\pi^{\otimes n} : U(\mathfrak{gl}_N) \rightarrow M$ is defined by

$$\pi^{\otimes n}(X) = \sum_{i=1}^n \text{Id}^{\otimes i-1} \otimes \pi(X) \otimes \text{Id}^{\otimes n-i}.$$

In general, any Lie group G acts on its Lie algebra \mathfrak{g} via the adjoint action

$$\text{Ad}(g)x = gxg^{-1}, \quad g \in G, x \in \mathfrak{g}.$$

This action extends naturally to $U(\mathfrak{gl}_N)$. For a subgroup K of G , let $U(\mathfrak{gl}_N)^K = \{x \in U(\mathfrak{gl}_N) : \text{Ad}(g)x = x \text{ for all } g \in K\}$. In particular, $U(\mathfrak{gl}_N)^G = Z(U(\mathfrak{gl}_N))$, the centre of $U(\mathfrak{gl}_N)$.

Recall that the Harish–Chandra isomorphism identifies $Z(U(\mathfrak{gl}_N))$ with shifted symmetric polynomials. Explicitly, each $X \in Z(U(\mathfrak{gl}_N))$ acts as some constant $p_X(\lambda)$ on the irreducible representation V_λ . It turns out that p_X is symmetric in the shifted variables $\lambda_i - i$.

2.2 Non-commutative probability

A non-commutative probability space (\mathcal{A}, ϕ) is a unital $*$ -algebra \mathcal{A} with identity 1 and a state $\phi : \mathcal{A} \rightarrow \mathbb{C}$, that is, a linear map such that $\phi(a^*a) \geq 0$ and $\phi(1) = 1$. Elements of \mathcal{A} are called *non-commutative random variables*. This generalises a classical probability space, by considering $\mathcal{A} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ with $\phi(X) = \mathbb{E}_\mathbb{P}X$. We also need a notion of convergence. For a large parameter L and $a_1, \dots, a_r \in \mathcal{A}, \phi$ which depend on L , as well as a limiting space (\mathbb{A}, Φ) , we say that (a_1, \dots, a_r) converges to $(\mathbf{a}_1, \dots, \mathbf{a}_r)$ with respect to the state ϕ if

$$\phi(a_{i_1}^{\epsilon_1} \dots a_{i_k}^{\epsilon_k}) \rightarrow \Phi(\mathbf{a}_{i_1}^{\epsilon_1} \dots \mathbf{a}_{i_k}^{\epsilon_k})$$

for any $i_1, \dots, i_k \in \{1, \dots, r\}, \epsilon_j \in \{1, *\}$ and $k \geq 1$.

There is also a non-commutative version of a Markov chain. If $X_n : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow E$ denotes the Markov process with transition operator $Q : L^\infty(E) \rightarrow L^\infty(E)$, then the Markov property is

$$\mathbb{E}[Yf(X_{n+1})] = \mathbb{E}[YQf(X_n)]$$

for $f \in L^\infty(E)$ and Y a $\sigma(X_1, \dots, X_n)$ -measurable random variable. Letting $j_n : L^\infty(E) \rightarrow L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ be defined by $j_n(f) = f(X_n)$, we can write the Markov property as

$$\mathbb{E}[j_{n+1}(f)Y] = \mathbb{E}[j_n(Qf)Y]$$

for all $f \in L^\infty(E)$ and Y in the subalgebra of $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ generated by the images of j_0, \dots, j_n .

Translating into the non-commutative setting, we define a *non-commutative Markov operator* to be a semigroup of completely positive unital linear

maps $\{P_t : t \in T\}$ from a $*$ -algebra U to itself (not necessarily an algebra morphism). The set T indexing time can be either \mathbb{N} or $\mathbb{R}_{\geq 0}$, that is to say, the Markov process can be either discrete or continuous time. If for any times $t_0 < t_1 < \dots \in T$ there exists algebra morphisms j_{t_n} from U to a non-commutative probability space (W, ω) such that

$$\omega(j_{t_n}(f)w) = \omega(j_{t_{n-1}}(P_{t_n-t_{n-1}}f)w)$$

for all $f \in U$ and w in the subalgebra of W generated by the images of $\{j_t : t \leq t_{n-1}\}$, then j_t is called a *dilation* of P_t .

3 Non-commutative random walk on $U(\mathfrak{gl}_N)$

The first thing that needs to be done is to define states on $U(\mathfrak{gl}_N)$. Note that is already has a natural $*$ -algebra structure,

Given any positive type, normalized (sending the identity to 1), class function $\kappa \in L^2(G)$, we have the decomposition

$$\kappa = \sum_{\lambda \in \widehat{G}} \widehat{\kappa}(\lambda) \frac{\chi_\lambda}{\dim \lambda},$$

where \widehat{G} denotes the set of equivalence classes of irreducible representations of G , and χ_λ are the characters corresponding to λ . By the orthogonality relations, $\widehat{\chi_\lambda}(\lambda) = 1$. This defines a state κ on M by

$$\kappa(X) = \sum_{\lambda} \widehat{\kappa}(\lambda) \sum_{i=1}^{\dim \lambda} \text{Tr}(X|_{V_\lambda^{(i)}}), \quad X \in M.$$

This naturally pulls back via $\pi : U(\mathfrak{gl}_N) \rightarrow M$ to a state $\kappa(\cdot)$ on $U(\mathfrak{gl}_N)$.

Recall an equivalent definition of these states from [6]. There is a canonical isomorphism $D : U(\mathfrak{gl}(N)) \rightarrow \mathcal{D}(N)$ where $\mathcal{D}(N)$ is the algebra of left-invariant differential operators on $U(N)$ with complex coefficients. Then the state $\langle \cdot \rangle_\kappa$ on $U(\mathfrak{gl}(N))$ is defined by

$$\langle X \rangle_\kappa = D(X)\kappa(U)|_{U=I}.$$

The state can be computed using the formula (see e.g. page 101 of [19]) for $X = E_{i_1 j_1} \cdots E_{i_k j_k}$:

$$D(X)\kappa(U) = \partial_{t_1} \cdots \partial_{t_k} \kappa \left(U e^{t_1 E_{i_1 j_1}} \cdots e^{t_k E_{i_k j_k}} \right) \Big|_{t_1=\dots=t_k=0}. \quad (2)$$

Comparing (2) and (1) shows that $D = \pi$. Here, $e^{t E_{ij}}$ is just the usual exponential of matrices, which has the simple expression

$$e^{t E_{ij}} = \begin{cases} Id + t E_{ij}, & i \neq j \\ Id + (e^t - 1) E_{ii} & i = j \end{cases} \quad (3)$$

Note that since (2) only involves linear terms in the t_j , one can replace $e^{t E_{ii}}$ with $Id + t E_{ii}$ without changing the value of the right hand side

of (2). This is a slightly different approach from [6], which used the (equivalent) formula

$$E_{ij} \mapsto \sum_k x_{ik} \partial_{jk}.$$

It is not hard to see that the two definitions of $\langle \cdot \rangle_\kappa$ are equivalent. For each $\lambda \in \widehat{G}$ and $X = E_{i_1 j_1} \cdots E_{i_k j_k}$, and letting v_1, \dots, v_d be a basis of V_λ ,

$$\begin{aligned} \langle X \rangle_{\chi_\lambda} &= \partial_{t_1} \cdots \partial_{t_k} \chi_\lambda \left(e^{t_1 E_{i_1 j_1}} \cdots e^{t_k E_{i_k j_k}} \right) \Big|_{t_1 = \dots = t_k = 0} \\ &= \partial_{t_1} \cdots \partial_{t_k} \sum_{r=1}^d \left\langle e^{t_1 E_{i_1 j_1}} \cdots e^{t_k E_{i_k j_k}} v_r, v_r \right\rangle \Big|_{t_1 = \dots = t_k = 0} \\ &= \sum_{r=1}^d \left\langle E_{i_1 j_1} \cdots E_{i_k j_k} v_r, v_r \right\rangle \\ &= \text{Tr} \left(X|_{V_\lambda} \right) \end{aligned}$$

By linearity, this holds for all κ and all X .

Now that the states have been defined, we define the non-commutative Markov process. In order to define a continuous-time non-commutative Markov process, there needs to be a semigroup $\{\kappa_t : t \geq 0\}$ in $L^2(G)$. Indeed, such a semigroup exists: for any $t \geq 0$, let

$$\kappa_t(U) = e^{t \text{Tr}(U - \text{Id})}.$$

Now fix times $t_1 < t_2 < \dots$. Let \mathcal{W} be the infinite tensor product of von Neumann algebras $M^{\otimes \infty}$ with respect to the state $\omega = \kappa_{t_1} \otimes \kappa_{t_2 - t_1} \otimes \kappa_{t_3 - t_2} \otimes \dots$. For $n \geq 1$ define the morphism $j_{t_n} : U(\mathfrak{gl}_N) \rightarrow \mathcal{W}$ to be the map $j_{t_n}(X) = \pi^{\otimes n}(X) \otimes \text{Id}^{\otimes \infty}$, and let \mathcal{W}_n be the subalgebra generated by the images of j_{t_1}, \dots, j_{t_n} . Define $P_t : U(\mathfrak{gl}_N) \rightarrow U(\mathfrak{gl}_N)$ by $(\text{id} \otimes \kappa_t) \circ \Delta$. Note that P is linear as a map of complex vector spaces, but is not an algebra morphism, since the trace is not preserved under multiplication of matrices. To simplify notation, write $\langle \cdot \rangle_t$ for $\langle \cdot \rangle_{\kappa_t}$ and j_n for j_{t_n} .

Theorem 3.1. (1) The maps (j_n) are a dilation of the non-commutative Markov operator P_t . In other words,

$$\omega(j_n(X)w) = \omega(j_{n-1}(P_{t_n - t_{n-1}}X)w), \quad X \in U(\mathfrak{gl}(N)), \quad w \in \mathcal{W}_{n-1}.$$

(2) The pullback of ω under j_n is the state $\langle \cdot \rangle$ on $U(\mathfrak{gl}(N))$, i.e. $\langle X \rangle_{t_n} = \omega(j_n(X))$.

(3) For $n \leq m$, we have

$$\omega(j_n(X)j_m(Y)) = \langle X \cdot P_{t_m - t_n}Y \rangle_{t_n}.$$

(4) The non-commutative markov operators P_t satisfy the semi-group property $P_{t+s} = P_t \circ P_s$.

(5) For any subgroup $K \subset U(N)$, the restriction of P_t to $U(\mathfrak{gl}_N)^K$ is still a non-commutative transition kernel. In other words, $P_t U(\mathfrak{gl}_N)^K \subset U(\mathfrak{gl}_N)^K$. In particular, $P_t Z(U(\mathfrak{gl}_N)) \subset Z(U(\mathfrak{gl}_N))$.

Proof. (1) This is essentially identical to Proposition 3.1 from [9]. It is reproduced here for completeness. The left-hand-side is

$$\begin{aligned}\omega((\pi^{\otimes n-1} \otimes \pi)\Delta X w) &= \sum_{(X)} \omega(\pi^{\otimes n-1}(X_{(1)}) \otimes \pi(X_{(2)})w) \\ &= \sum_{(X)} \omega(\pi^{\otimes n-1}(X_{(1)})w) \langle X_{(2)} \rangle_{t_n - t_{n-1}}.\end{aligned}$$

The right-hand-side is

$$\sum_{(X)} \omega(j_{n-1}(\langle X_{(2)} \rangle_{t_n - t_{n-1}} X_{(1)} w) = \sum_{(X)} \omega(j_{n-1}(X_{(1)})w) \langle X_{(2)} \rangle_{t_n - t_{n-1}}.$$

(2) Let m_n denote the n -fold multiplication $L^2(G)^{\otimes n} \rightarrow L^2(G)$ that sends $f_1 \otimes \cdots \otimes f_n$ to $f_1 \cdots f_n$. We will show that

$$m_n(\pi^{\otimes n}(X)(f_1 \otimes \cdots \otimes f_n)) = \pi(X)(f_1 \cdots f_n). \quad (4)$$

The case of general n follows inductively from $n = 2$. The left hand side is

$$\sum_{(X)} (\pi(X^{(1)})f_1) \cdot (\pi(X^{(1)})f_2) = \sum_{(X, f_1, f_2)} \langle X^{(1)}, f_1^{(2)} \rangle f_1^{(1)} \cdot \langle X^{(2)}, f_2^{(2)} \rangle f_2^{(1)}.$$

The right hand side is

$$\langle \text{id} \otimes X, \Delta(f_1 \cdot f_2) \rangle = \sum_{(f_1, f_2)} \langle X, f_1^{(2)} f_2^{(2)} \rangle f_1^{(1)} \cdot f_2^{(1)}.$$

So it suffices to show that

$$\sum_{(X)} \langle X^{(1)}, f_1^{(2)} \rangle \langle X^{(2)}, f_2^{(2)} \rangle = \langle X, f_1^{(2)} f_2^{(2)} \rangle.$$

But this is just the definition of multiplication in a dual Hopf algebra. So (4) is true.

Now recall that if A is a Hopf algebra with co-unit $\epsilon : A \rightarrow \mathbb{C}$, then the n -fold tensor power $A^{\otimes n}$ is also a Hopf algebra, with co-unit $\epsilon^{(n)} : A^{\otimes n} \rightarrow \mathbb{C}$ defined by the composition

$$A^{\otimes n} \xrightarrow{m_n} A \xrightarrow{\epsilon} \mathbb{C}.$$

In other words,

$$\epsilon^{(n)}(a_1 \otimes \cdots \otimes a_n) = \epsilon(a_1 \cdots a_n) = \epsilon(a_1) \cdots \epsilon(a_n).$$

The second equality holds because ϵ is a morphism of \mathbb{C} -algebras.

When $A = L^2(G)$, then $\epsilon : L^2(G) \rightarrow \mathbb{C}$ is defined by $\epsilon(f) = f(\text{Id}_G)$. I now claim that for $X \in U(\mathfrak{gl}_N)$

$$\omega(\pi^{\otimes n}(X)) = \epsilon^{(n)} X(\kappa_{t_n}). \quad (5)$$

For $n = 1$, this follows immediately from the definitions. For $n \geq 2$, write as usual

$$\pi^{\otimes n}(X) = \sum_{(X)} X_{(1)} \otimes \cdots \otimes X_{(n)} \in M^{\otimes n}.$$

Then

$$\begin{aligned}\omega(\pi^{\otimes n}(X)) &= \sum_{(X)} \kappa_{t_1}(X_{(1)}) \cdots \kappa_{t_n-t_{n-1}}(X_{(n)}) \\ &= \sum_{(X)} \epsilon X_{(1)}(\kappa_{t_1}) \cdots \epsilon X_{(n)}(\kappa_{t_n-t_{n-1}}).\end{aligned}$$

At the same time,

$$\begin{aligned}\epsilon^{(n)} X(\kappa_{t_n}) &= \epsilon^{(n)} X(\kappa_{t_1} \cdots \kappa_{t_n-t_{n-1}}) \\ &= \epsilon^{(n)} \sum_{(X)} X_{(1)}(\kappa_{t_1}) \otimes \cdots \otimes X_{(n)}(\kappa_{t_n-t_{n-1}}) \\ &= \sum_{(X)} \epsilon X_{(1)}(\kappa_{t_1}) \cdots \epsilon X_{(n)}(\kappa_{t_n-t_{n-1}}).\end{aligned}$$

So (5) is true.

Finally, we can combine the results to obtain

$$\begin{aligned}\omega(j_n(X)) &= \omega(\pi^{\otimes n}(X)) = \epsilon^{(n)} \pi^{\otimes n} X(\kappa_{t_1} \otimes \cdots \otimes \kappa_{t_n-t_{n-1}}) \\ &= \epsilon m_n(\pi^{\otimes n} X(\kappa_{t_1} \otimes \cdots \otimes \kappa_{t_n-t_{n-1}})) \\ &= \epsilon \pi(X)(\kappa_{t_1} \otimes \cdots \otimes \kappa_{t_n-t_{n-1}}) = \langle X \rangle_{\kappa_{t_n}}.\end{aligned}$$

This is exactly what (2) stated.

(3) By repeated applications of (1),

$$\omega(j_n(X)j_m(Y)) = \omega(j_n(X)j_n(P_{t_m-t_n}Y)).$$

Since j_n is a morphism of algebras, this equals $\omega(j_n(X \cdot P_{t_m-t_n}Y))$, which by (2) equals the right-hand-side of (3).

(4) By linearity, it suffices to prove this for monomials of the form $E = E_{i_1 j_1} \cdots E_{i_k j_k}$. Introduce some notation: let $K = \{1, \dots, k\}$ and for any subset $S \subseteq K$, define $E_S = \prod_{s \in S} E_{i_s j_s}$, where the product is taken in increasing order. The term E_\emptyset is understood to be 1. So, for example, if $E = E_{13}E_{42}E_{55}E_{12}$ and $S = \{1, 2, 4\}$ then $E_S = E_{13}E_{42}E_{12}$. With this notation,

$$\Delta E = \sum_{S \subseteq K} E_S \otimes E_{K \setminus S}.$$

And therefore

$$\begin{aligned}P_{t_2-t_1}E &= \sum_{S \subseteq K} \langle E_{K \setminus S} \rangle_{t_2-t_1} E_S, \\ P_{t_1}P_{t_2-t_1}E &= \sum_{\substack{S \subseteq K \\ R \subseteq S}} \langle E_{S \setminus R} \rangle_{t_1} \langle E_{K \setminus S} \rangle_{t_2-t_1} E_R.\end{aligned}$$

Since

$$P_{t_2}E = \sum_{R \subseteq K} \langle E_{K \setminus R} \rangle_{t_2} E_R,$$

it suffices to show

$$\langle E_{K \setminus R} \rangle_{t_2} = \sum_{R \subseteq S \subseteq K} \langle E_{K \setminus S} \rangle_{t_2-t_1} \langle E_{S \setminus R} \rangle_{t_1} \quad \text{for all } R \subseteq K,$$

or equivalently

$$\langle E_K \rangle_{t_2} = \sum_{S \subseteq K} \langle E_{K \setminus S} \rangle_{t_2 - t_1} \langle E_S \rangle_{t_1}.$$

This follows from (2) and the general Leibniz rule applied to the derivatives of the product $\kappa_{t_1} \cdot \kappa_{t_2 - t_1}$.

(5) This is Proposition 4.3 from [9]. Here is also a bare-bones proof when $K = G$. Let $X \in Z(U(\mathfrak{gl}_N))$. The goal is to show that $P(X)Y = YP(X)$ for all $Y \in U(\mathfrak{gl}_N)$. It suffices to show this when $Y \in \mathfrak{g}$. In this case,

$$\begin{aligned} XY = YX &\implies \Delta(XY) = \Delta(YX) \implies \Delta(X)\Delta(Y) = \Delta(Y)\Delta(X) \\ &\implies \sum_{(X)} X_{(1)}Y \otimes X_{(2)} + X_{(1)} \otimes X_{(2)}Y = \sum_{(X)} YX_{(1)} \otimes X_{(2)} + X_{(1)} \otimes YX_{(2)}. \end{aligned}$$

Now apply the linear map $id \otimes \langle \cdot \rangle$ to both sides to get

$$\sum_{(X)} \langle X_{(2)} \rangle X_{(1)}Y + \langle X_{(2)}Y \rangle X_{(1)} = \sum_{(X)} \langle X_{(2)} \rangle YX_{(1)} + \langle YX_{(2)} \rangle X_{(1)}.$$

Since the state $\langle \cdot \rangle$ is tracial, the second summand on both sides are equal. The first summand on the left-hand-side is $P(X)Y$ while the first summand on the right-hand-side is $YP(X)$, so $P(X) \in Z(U(\mathfrak{gl}_N))$ as needed. \square

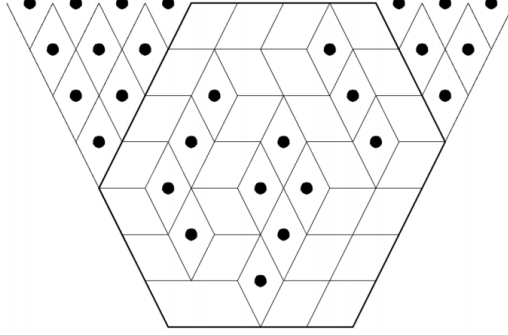
4 Connections to classical probability

In this section, we will show that restricting to the centres $Z(U(\mathfrak{gl}_1), \dots, Z(U(\mathfrak{gl}_N)))$ reduces the non-commutative random walk to a (2+1)-dimensional random surface growth model. First, here is a description of the model, which was introduced in [7].

4.1 Random surface growth

Consider the two-dimensional lattice $\mathbb{Z} \times \mathbb{Z}_+$. On each horizontal level $\mathbb{Z} \times \{n\}$ there are exactly n particles, with at most one particle at each lattice site. Let $X_1^{(n)} > \dots > X_n^{(n)}$ denote the x -coordinates of the locations of the n particles. Additionally, the particles need to satisfy the *interlacting property* $X_{i+1}^{(n+1)} < X_i^{(n)} \leq X_i^{(n+1)}$. The particles can be viewed as a random stepped surface, see Figure 1. This can be made rigorous by defining the height function at (x, n) to be the number of particles to the right of (x, n) .

Figure 1: The particles as a stepped surface. The lattice is shifted to make the visualization easier.



The dynamics on the particles are as follows. The initial condition is the *densely packed* initial condition, $\Lambda_i^{(n)} = -i + 1, 1 \leq i \leq n$. Each particle has a clock with exponential waiting time of rate 1, with all clocks independent of each other. When the clock rings, the particle attempts to jump one step to the right. However, it must maintain the interlacing property. This is done by having particles push particles above it, and jumps are blocked by particles below it. One can think of lower particles as being more massive. See Figure 2 for an example.

The projection to $\mathbb{Z} \times \{n\}$ is still Markovian, and is known as the *Charlier process* [14]. It can be described by a continuous-time Markov chain on \mathbb{Z}^n with independent increments $e_i/n, 1 \leq i \leq n$, (where $\{e_i\}$ is the canonical basis for \mathbb{Z}^n) conditioned to stay in the Weyl chamber $(x_1 > x_2 > \dots > x_n)$. Equivalently, the conditioned Markov chain is the Doob h -transform for some harmonic function h . There is a nice description of h in terms of representation theory, namely, $h(x_1, \dots, x_n)$ is the dimension of the irreducible representation of \mathfrak{gl}_n with highest weight $(x_1, x_2 + 1, \dots, x_n + n - 1)$. Explicitly,

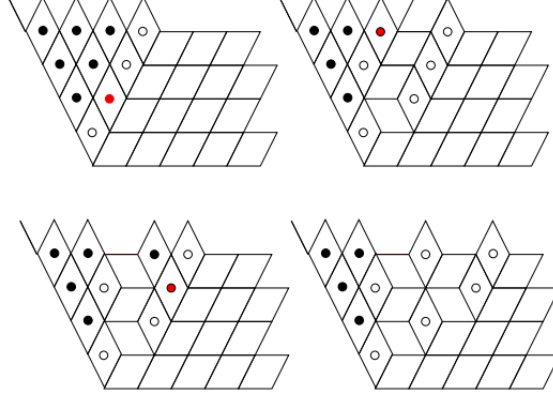
$$\dim \lambda = \prod_{i < j} \frac{\lambda_i - i - (\lambda_j - j)}{j - i}.$$

Below, let $Q_t^{(N)}$ denote the Markov operator of this Markov chain.

The construction of the full particle system is based on a general multivariate construction from [7], which is based on [10]. Suppose there are two Markov chains with state spaces $\mathcal{S}, \mathcal{S}^*$ and transition probabilities P, P^* . Also assume there is a Markov operator $\Lambda : \mathcal{S}^* \rightarrow \mathcal{S}$ which intertwines with P, P^* in the sense that $\Lambda P^* = P\Lambda$. In other words, there is a commutative diagram

$$\begin{array}{ccc} \mathcal{S}^* & \xrightarrow{P^*} & \mathcal{S}^* \\ \downarrow \Lambda & & \downarrow \Lambda \\ \mathcal{S} & \xrightarrow{P} & \mathcal{S} \end{array} \quad (6)$$

Figure 2: The red particle makes a jump. If any of the black particles attempt to jump, their jump is blocked by the particle below and to the right, and nothing happens. White particles are not blocked.



Then the state space is $\{(x^*, x) \in \mathcal{S}^* \times \mathcal{S} : \Lambda(x^*, x) \neq 0\}$ with transition probabilities

$$\text{Prob}((x^*, x) \rightarrow (y^*, y)) = \begin{cases} \frac{P(x, y) P^*(x^*, y^*) \Lambda(y^*, y)}{\Delta(x^*, y)}, & \Delta(x^*, y) \neq 0 \\ 0, & \Delta(x^*, y) = 0 \end{cases}$$

Additionally, if the initial condition is a *Gibbs measure*, that is, a probability distribution of the form $\mathbb{P}(x^*) \Lambda(x^*, x)$, then the dynamics preserves Gibbs measures. All constructions and definitions extend naturally to any finite number of Markov chains.

Here, $Q_t^{(N)}$ and $Q_t^{(N-1)}$ will play the roles of P^* , P , and the projection Λ is

$$\Lambda(x_1 > \dots > x_N, y_1 > \dots > y_{N-1}) = \frac{h(y)}{h(x)}.$$

The construction implies that

$$\mathbb{P}(X^{(N)}(t) = x^{(N)} | X^{(M)}(t) = x^{(M)}) = \frac{h(x^{(N)})}{h(x^{(M)})}, \forall N \leq M, t \geq 0 \quad (7)$$

$$\begin{aligned} \mathbb{P}(X^{(N)}(t) = x^{(N)} | X^{(M)}(s) = y^{(M)}, X^{(N)}(s) = y^{(N)}) \\ = \mathbb{P}(X^{(N)}(t) = x^{(N)} | X^{(N)}(s) = y^{(N)}), \forall N \leq M, s \leq t \end{aligned} \quad (8)$$

The intuition behind (8) is that since particles on lower levels push and block the particles on higher levels, the evolution of the N -th level is independent of the evolution M -th level. Equation (7) is a mathematical formulation of the statement that the dynamics preserves Gibbs measures.

4.2 Restriction to centre

Before continuing, we need to compute the states of certain observables.

Proposition 4.1. *Let Π denote the set of partitions of the set $\{1, \dots, m\}$, let $|\pi|$ denote the number of blocks of the partition $\pi \in \Pi$ and let $B \in \pi$ mean that B is a block in π . Then*

$$\langle E_{i_1 j_1} \cdots E_{i_m j_m} \rangle_t = \sum_{\pi \in \Pi} t^{|\pi|} \prod_{\substack{B \in \pi \\ B = \{b_1, \dots, b_k\}}} 1_{j_{b_1} = i_{b_2}, j_{b_2} = i_{b_3}, \dots, j_{b_k} = i_{b_1}}$$

Example 1

$$\langle E_{21} E_{12} E_{21} E_{12} \rangle_t = 2t^2 + t$$

with two contributing partitions having two blocks: $\{1, 2\} \cup \{3, 4\}$, $\{1, 4\} \cup \{2, 3\}$, and one contributing partition having one block $\{1, 2, 3, 4\}$.

Example 2

$$\langle E_{11}^3 E_{22} \rangle_t = t^4 + 3t^3 + t^2$$

with one contributing partition having four blocks: $\{1\} \cup \{2\} \cup \{3\} \cup \{4\}$, three contributing partitions having three blocks: $\{1, 2\} \cup \{3\} \cup \{4\}$, $\{1, 3\} \cup \{2\} \cup \{4\}$, $\{2, 3\} \cup \{1\} \cup \{4\}$, and one contributing partition having two blocks: $\{1, 2, 3\} \cup \{4\}$.

Example 3 For any m ,

$$\langle E_{jj}^m \rangle_t = B_m(t)$$

where $B_m(t)$ is the m -th Bell polynomial. These are also the moments of a Poisson random variable with mean t , so under the state $\langle \cdot \rangle_t$, each E_{jj} can be heuristically understood to be distributed as Poisson(t).

Example 4

$$\langle E_{11} E_{12} \rangle_t = 0$$

with no contributing partitions.

Proof. Recall the definition of $\langle \cdot \rangle_t$ in (2). By Faa di Bruno formula,

$$\langle E_{i_1 j_1} \cdots E_{i_m j_m} \rangle_t = \sum_{\pi \in \Pi} f^{(|\pi|)}(y) \prod_{B \in \pi} \frac{\partial^{|B|} y}{\prod_{b \in B} \partial x_b} \Big|_{x_1 = \dots = x_m = 0}$$

where

$$f(y) = e^{ty}, \quad y = \text{Tr}(e^{x_1 E_{i_1 j_1}} \cdots e^{x_m E_{i_m j_m}} - \text{Id}).$$

Note that

$$f^{(|\pi|)}(y) \Big|_{x_1 = \dots = x_m = 0} = t^{|\pi|} f(y) \Big|_{y=0} = t^{|\pi|}.$$

Since we are taking the derivative with respect to x_b and setting equal to 0, we only need the linear terms in x_b , so it is equivalent to replace y with

$$y = \text{Tr}((\text{Id} + x_1 E_{i_1 j_1}) \cdots (\text{Id} + x_m E_{i_m j_m}) - \text{Id}).$$

Here, E_{ij} are the usual $N \times N$ matrices acting on \mathbb{C}^N , not the generators of $U(\mathfrak{gl}_N)$. Expanding the parantheses, all terms other than

$Tr\left(\prod_{b \in B} x_b E_{i_b j_b}\right)$ do not contribute, since these do not survive differentiation with respect to $x_b, b \in B$. Finally, since

$$Tr\left(\prod_{\substack{b \in B \\ B=\{b_1, \dots, b_k\}}} E_{i_b j_b}\right) = 1_{j_{b_1}=i_{b_2}, j_{b_2}=i_{b_3}, \dots, j_{b_k}=i_{b_1}},$$

the proof is finished. \square

In section 7 of [11], explicit generators of the centre $Z(U(\mathfrak{gl}_N))$ were found. See also chapter 7 of [16] for an exposition.

Let \mathcal{G}_m denote the directed graph with vertices and edges

$$\{1, \dots, m\} \quad \{(i, j) : 1 \leq i, j \leq m\}.$$

Let $\Pi_k^{(m)}$ denote the set of all paths in \mathcal{G}_m of length k which start and end at the vertex m . For $\pi \in \Pi_k^{(m)}$ let $r(\pi)$ denote the length of the first return to m . Let $E(\pi) \in U(\mathfrak{gl}_m)$ denote the element with coefficient $r(\pi)$ obtained by taking the product when labeling the edge (i, j) with E_{ij} when $i \neq j$, and the edge (i, i) with $E_{ii} - m + 1$. For example, the path

$$\pi = \{5 \rightarrow 3 \rightarrow 3 \rightarrow 1 \rightarrow 5 \rightarrow 5 \rightarrow 2 \rightarrow 5\}$$

is in $\Pi_7^{(5)}$ with $r(\pi) = 4$ and

$$E(\pi) = 4E_{53}(E_{33} - 4)E_{31}E_{15}(E_{55} - 4)E_{52}E_{25}.$$

Define the elements

$$\Psi_k := \sum_{m=1}^N \sum_{\pi \in \Pi_k^{(m)}} E(\pi) \in U(\mathfrak{gl}_N).$$

For example,

$$\Psi_1 = \sum_{m=1}^N (E_{mm} - m + 1), \quad \Psi_2 = \sum_{m=1}^N (E_{mm} - m + 1)^2 + 2 \sum_{1 \leq l < m \leq N} E_{ml} E_{lm}.$$

When we wish to emphasize that $\Psi_k \in U(\mathfrak{gl}_N)$, the notation $\Psi_k^{(N)}$ will be used.¹

Theorem 4.2. [11] *The centre $Z(U(\mathfrak{gl}_N))$ is generated by the elements $1, \{\Psi_k\}_{k \geq 1}$. Furthermore, the Harish-Chandra isomorphism maps Ψ_k to the shifted symmetric polynomial $\sum_{m=1}^N (\lambda_m - m + 1)^k$.*

Remark. Writing $\mathfrak{gl}_N = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ where $\mathfrak{n}_+, \mathfrak{n}_-$ are the upper and lower nilpotent subalgebras and \mathfrak{h} is the diagonal subalgebra, the Harish-Chandra homomorphism is the projection

$$U(\mathfrak{gl}_N) = (\mathfrak{n}_- U(\mathfrak{gl}_N) + U(\mathfrak{gl}_N) \mathfrak{n}_+) \oplus U(\mathfrak{h}) \rightarrow U(\mathfrak{h}) = S(\mathfrak{h}) = \mathbb{C}[\lambda_1, \dots, \lambda_N].$$

¹Caution: This notation is consistent with notation from integrable probability but different from notation in representation theory.

This sends

$$\Psi_k = \sum_{m=1}^N (E_{mm} - m + 1)^k + (\text{other terms}) \mapsto \sum_{m=1}^N (E_{mm} - m + 1)^k = \sum_{m=1}^N (\lambda_m - m + 1)^k.$$

Of course, $\sum_{m=1}^N (E_{mm} - m + 1)^k$ is in general not central.

Now it is time to explicitly state the relationship between the non-commutative random walk and the growing stepped surface. One may be tempted to think that

$$\begin{array}{ccc} U(\mathfrak{gl}_N) & \xrightarrow{P_t} & U(\mathfrak{gl}_N) \\ \uparrow & & \uparrow \\ U(\mathfrak{gl}_{N-1}) & \xrightarrow{P_t} & U(\mathfrak{gl}_{N-1}) \end{array}$$

is a non-commutative version of (6). However, care needs to be taken because the inclusion map does not send $Z(U(\mathfrak{gl}_{N-1}))$ to $Z(U(\mathfrak{gl}_N))$.

A slight change of variables will make statements cleaner. If $p(\lambda)$ is a shifted symmetric polynomial, then by definition it is symmetric in the variables $x_i = \lambda_i - i + 1$, and let $\bar{p}(x)$ denote the corresponding symmetric polynomial.

Proposition 4.3. *If $Y \in Z(U(\mathfrak{gl}_N))$ is sent to the symmetric polynomial $p_Y(x)$ by the Harish-Chandra isomorphism, then*

$$\langle Y \rangle_t = \mathbb{E} \left[\bar{p}_Y(X_1^{(N)}(t), \dots, X_N^{(N)}(t)) \right].$$

Proof. This is not new, see [6], but the proof is similar to Theorem 4.5 below, so will be repeated for clarity. By a result from [8],

$$e^{t\text{Tr}(U - \text{Id})} = \sum_{\lambda} \text{Prob}(X_i^{(N)}(t) = \lambda_i - i + 1, 1 \leq i \leq N) \frac{\chi_{\lambda}(U)}{\dim \lambda}$$

where χ_{λ} and $\dim \lambda$ are the character and dimension of the highest weight representation λ . Thus, by linearity,

$$\begin{aligned} \langle Y \rangle_t &= \sum_{\lambda} \text{Prob}(X_i^{(N)}(t) = \lambda_i - i + 1, 1 \leq i \leq N) \frac{\langle Y \rangle_{\chi_{\lambda}}}{\dim \lambda} \\ &= \sum_{\lambda} \text{Prob}(X_i^{(N)}(t) = \lambda_i - i + 1, 1 \leq i \leq N) p_Y(\lambda_1, \dots, \lambda_N) \\ &= \sum_x \text{Prob}(X_i^{(N)}(t) = x_i, 1 \leq i \leq N) \bar{p}_Y(x_1, \dots, x_N) \end{aligned}$$

The last line is simply the right-hand-side of the proposition. \square

Proposition 4.4. *Suppose that P_t and Q_t are two semigroups which preserve $Z(U(\mathfrak{gl}_N))$ and satisfy Theorem 3.1(1), then $P_t X = Q_t X$ for all $X \in Z(U(\mathfrak{gl}_N))$. In particular, P_t is the Markov operator of the process $(X_1^{(N)}(t) > \dots > X_N^{(N)}(t))$.*

One can check explicitly that the semigroup property holds.

Example 7 We wish to take asymptotics $N \approx \eta L$ and $t \approx \tau L$. We would get

$$\begin{aligned} P_t \Psi_{1,N} &= \Psi_1 + \text{const}, \\ P_t \Psi_{2,N} &= \Psi_2 + 2\tau L \Psi_1 + \text{const}, \\ P_t \Psi_{3,N} &= \Psi_3 + 3\tau L \Psi_2 + 3(\tau^2 + \eta\tau)L^2 \Psi_1 + \text{const} \\ P_t \Psi_{4,N} &= \Psi_4 + 4\tau L \Psi_3 + (6\tau^2 + 4\eta\tau)L^2 \Psi_2 + (4\tau^3 + 12\tau^2\eta + 2\tau\eta^2)L^3 \Psi_1 \\ &\quad + 2\tau L \Psi_1^2 + \text{const} \\ P_t \Psi_{1,N}^2 &= \Psi_{1,N}^2 + 2\eta\tau L^2 \Psi_{1,N} + \text{const} \end{aligned}$$

Again, one can check that the semigroup property holds.

Theorem 4.5. Suppose $Y_1 \in Z(U(\mathfrak{gl}_{N_1})), \dots, Y_r \in Z(U(\mathfrak{gl}_{N_r}))$ are mapped to the symmetric polynomials $\bar{p}_{Y_1}, \dots, \bar{p}_{Y_r}$ under the Harish-Chandra isomorphism. Assume that $N_1 \geq \dots \geq N_r$ and $t_1 \leq \dots \leq t_r$. Then

$$\langle Y_1 P_{t_2-t_1} Y_2 \cdots P_{t_r-t_1} Y_r \rangle_{t_1} = \mathbb{E} \left[\bar{p}_{Y_1}(X^{(N_1)}(t_1)) \cdots \bar{p}_{Y_r}(X^{(N_r)}(t_r)) \right].$$

Proof. In order to simplify notation and elucidate the idea of the proof, assume $r = 2$. The more general case follows from exactly the same argument.

First prove it for $t_1 = t_2$. Assume $N_1 = N \geq M = N_2$. Let $m(\lambda, \mu)$ denote the multiplicity of μ in the restricted representation $V_\lambda|_{U(M)}$. Use $\bar{m}(\cdot, \cdot)$ to denote the same quantity in the shifted co-ordinates $x_i = \lambda_i - i + 1$. Then by the Gibbs property, that is (7),

$$\begin{aligned} \text{RHS} &= \sum_{x^{(N)}, x^{(M)}} \text{Prob}(X^{(N)}(t) = x_i^{(N)}, X^{(M)} = x_j^{(M)}) \bar{p}_{Y_1}(x^{(N)}) \bar{p}_{Y_2}(x^{(M)}) \\ &= \sum_{x^{(N)}, x^{(M)}} \text{Prob}(X^{(N)}(t) = x_i^{(N)}) \frac{\bar{m}(x^{(N)}, x^{(M)}) h(x^{(M)})}{h(x^{(N)})} \bar{p}_{Y_1}(x^{(N)}) \bar{p}_{Y_2}(x^{(M)}) \end{aligned}$$

At the same time,

$$\langle Y_1 Y_2 \rangle_t = \sum_{\lambda^{(N)}} \text{Prob}(X^{(N)}(t) = \lambda_i^{(N)} - i + 1) \frac{1}{\dim \lambda^{(N)}} \text{Tr}(Y_1 Y_2|_{V_{\lambda^{(N)}}}).$$

Since Y_1 is central, it acts as $p_{Y_1}(\lambda^{(N)}) \text{Id}$ on $V_{\lambda^{(N)}}$, so this equals

$$\sum_{\lambda^{(N)}} \text{Prob}(X^{(N)}(t) = \lambda_i^{(N)} - i + 1) \frac{p_{Y_1}(\lambda^{(N)})}{\dim \lambda^{(N)}} \text{Tr}(Y_2|_{V_{\lambda^{(N)}}}).$$

By restricting $V_{\lambda^{(N)}}$ to $U(M)$ and using that Y_2 acts as $p_{Y_2}(\lambda^{(M)}) \text{Id}$ on $V_{\lambda^{(M)}}$, we get

$$\sum_{\lambda^{(N)}, \lambda^{(M)}} \text{Prob}(X^{(N)}(t) = \lambda_i^{(N)} - i + 1) \frac{m(\lambda^{(N)}, \lambda^{(M)}) \dim \lambda^{(M)}}{\dim \lambda^{(N)}} p_{Y_1}(\lambda^{(N)}) p_{Y_2}(\lambda^{(M)}).$$

This is equal to the right-hand-side from above.

Now consider when $t = t_1 \leq t_2 = s$. Write $P_{s-t}Y_2$ as a sum over basis elements, that is $P_{s-t}Y_2 = \sum_{\rho} c_{\rho} Y_{\rho}$. Then

$$\begin{aligned} \langle Y_1 P_{s-t} Y_2 \rangle_t &= \sum_{\rho} c_{\rho} \langle Y_1 Y_{\rho} \rangle_t \\ &= \sum_{\rho} c_{\rho} \mathbb{E} \left[\bar{p}_{Y_1}(X^{(N)}(t)) \bar{p}_{Y_{\rho}}(X^{(M)}(t)) \right] \\ &= \mathbb{E} \left[\bar{p}_{Y_1}(X^{(N)}(t)) (P_{s-t} \bar{p}_{Y_2})(X^{(M)}(t)) \right] \end{aligned}$$

Thus, it suffices to prove that

$$\mathbb{E} \left[\bar{p}_{Y_1}(X^{(N)}(t)) \bar{p}_{Y_2}(X^{(M)}(s)) \right] = \mathbb{E} \left[\bar{p}_{Y_1}(X^{(N)}(t)) (P_{s-t} \bar{p}_{Y_2})(X^{(M)}(t)) \right]$$

We have

$$\begin{aligned} &\mathbb{E} \left[\bar{p}_{Y_1}(X^{(N)}(t)) \bar{p}_{Y_2}(X^{(M)}(s)) \right] \\ &= \sum_{y^{(N)}, x^{(M)}} \bar{p}_{Y_1}(y^{(N)}) \bar{p}_{Y_2}(x^{(M)}) \mathbb{P}(X^{(M)}(s) = x^{(M)}, X^{(N)}(t) = y^{(N)}) \\ &= \sum_{y^{(N)}, y^{(M)}, x^{(M)}} \bar{p}_{Y_1}(y^{(N)}) \bar{p}_{Y_2}(x^{(M)}) \mathbb{P}(X^{(M)}(s) = x^{(M)}, X^{(N)}(t) = y^{(N)}, X^{(M)}(t) = y^{(M)}) \\ &= \sum_{y^{(N)}, y^{(M)}, x^{(M)}} \bar{p}_{Y_1}(y^{(N)}) \bar{p}_{Y_2}(x^{(M)}) \mathbb{P}(X^{(M)}(s) = x^{(M)} | X^{(N)}(t) = y^{(N)}, X^{(M)}(t) = y^{(M)}) \\ &\quad \times \mathbb{P}(X^{(N)}(t) = y^{(N)}, X^{(M)}(t) = y^{(M)}) \end{aligned}$$

By (8) and the fact that $P_t = Q_t$, this then equals

$$\begin{aligned} &= \sum_{y^{(N)}, y^{(M)}, x^{(M)}} \bar{p}_{Y_1}(y^{(N)}) \bar{p}_{Y_2}(x^{(M)}) \mathbb{P}(X^{(M)}(s) = x^{(M)} | X^{(M)}(t) = y^{(M)}) \\ &\quad \times \mathbb{P}(X^{(N)}(t) = y^{(N)}, X^{(M)}(t) = y^{(M)}) \\ &= \sum_{y^{(N)}, y^{(M)}} \bar{p}_{Y_1}(y^{(N)}) (P_{s-t} \bar{p}_{Y_2})(y^{(M)}) \mathbb{P}(X^{(N)}(t) = y^{(N)}, X^{(M)}(t) = y^{(M)}) \\ &= \mathbb{E} \left[\bar{p}_{Y_1}(X^{(N)}(t)) (P_{s-t} \bar{p}_{Y_2})(X^{(M)}(t)) \right] \end{aligned}$$

□

We wrap up this section by giving an example showing that although $P_t = Q_t$ on $Z(U(\mathfrak{gl}_N))$, they are not equal on subalgebras generated by different $Z(U(\mathfrak{gl}_N))$. The determinantal formula from [7] yields

$$Q_1(\Psi_1^{(2)} \Psi_1^{(1)})(\lambda^{(2)}, \lambda^{(1)}) \approx 2.37 \dots, \text{ when } \lambda^{(2)} = (1, 0), \lambda^{(1)} = (0).$$

However,

$$P_t(\Psi_1^{(2)} \Psi_1^{(1)}) = \Psi_1^{(2)} \Psi_1^{(1)} + 2t \Psi_1^{(1)} + t \Psi_1^{(2)} + 2t^2 + t,$$

and when evaluated at $\lambda^{(2)} = (1, 0), \lambda^{(1)} = (0), t = 1$ yields 3.

5 Covariance Structure

In this section, it will be shown that the central elements are asymptotically Gaussian with an explicit covariance that generalizes the Gaussian free field. Let us review some previously known results.

Theorem 5.1. [6, 7] Suppose $N_j = \lfloor \eta_j L \rfloor, t_j = \tau_j L$ for $1 \leq j \leq r$. Assume they lie on a space-like path, that is $N_1 \geq \dots \geq N_r$ and $t_1 \leq \dots \leq t_r$. Then as $L \rightarrow \infty$,

$$\left(\frac{\Psi_{k_1}^{(N_1)} - \langle \Psi_{k_1}^{(N_1)} \rangle_{t_1}}{L^{k_1}}, \dots, \frac{P_{t_r-t_1} \Psi_{k_r}^{(N_r)} - \langle P_{t_r-t_1} \Psi_{k_r}^{(N_r)} \rangle_{t_1}}{L^{k_r}} \right) \rightarrow (\xi_1, \dots, \xi_r),$$

where the convergence is with respect to the state $\langle \cdot \rangle_{t_1}$, and (ξ_1, \dots, ξ_r) is a Gaussian vector with covariance

$$\mathbb{E}[\xi_i \xi_j] = \left(\frac{1}{2\pi i} \right)^2 \int \int_{|z| > |w|} (\eta_i z^{-1} + \tau_i + \tau_i z)^{k_i} (\eta_j w^{-1} + \tau_j + \tau_j w)^{k_j} (z-w)^{-2} dz dw.$$

The proof uses that the particle system is a determinantal point process along space-like paths. This condition is necessary due to the construction using (6). In particular, there are no maps going up from S to S^* . A natural question is to ask what happens along time-like paths, that is, $N_1 \leq N_2, t_1 \leq t_2$. The main theorem is

Theorem 5.2. Suppose $N_j = \lfloor \eta_j L \rfloor, t_j = \tau_j L$ for $1 \leq j \leq r$. Assume $\min(\tau_1, \dots, \tau_r) = \tau_1$. Then as $L \rightarrow \infty$

$$\left(\frac{\Psi_{k_1}^{(N_1)} - \langle \Psi_{k_1}^{(N_1)} \rangle_{t_1}}{L^{k_1}}, \dots, \frac{P_{t_r-t_1} \Psi_{k_r}^{(N_r)} - \langle P_{t_r-t_1} \Psi_{k_1}^{(N_1)} \rangle_{t_1}}{L^{k_r}} \right) \rightarrow (\xi_1, \dots, \xi_r),$$

where the convergence is with respect to the state $\langle \cdot \rangle_{t_1}$, and (ξ_1, \dots, ξ_r) is a Gaussian vector with covariance

$$\mathbb{E}[\xi_i \xi_j] = \begin{cases} \left(\frac{1}{2\pi i} \right)^2 \int \int_{|z| > |w|} (\eta_i z^{-1} + \tau_i + \tau_i z)^{k_i} (\eta_j w^{-1} + \tau_j + \tau_j w)^{k_j} (z-w)^{-2} dz dw, & \eta_i \geq \eta_j, \tau_i \leq \tau_j \\ \left(\frac{1}{2\pi i} \right)^2 \int \int_{|z| > |w|} (\eta_j \frac{\tau_j}{\tau_i} z^{-1} + \tau_j + \tau_i z)^{k_j} (\eta_i w^{-1} + \tau_i + \tau_i w)^{k_i} (z-w)^{-2} dz dw, & \eta_i < \eta_j, \tau_i \leq \tau_j \end{cases}$$

Example 8 The double integral can be computed using residues and the Taylor series

$$(z-w)^{-2} = z^{-2} \left(1 + 2\frac{w}{z} + 3\frac{w^2}{z^2} + \dots \right).$$

So for instance,

$$\left\langle \left(\frac{\Psi_1^{(\eta_1 L)} - \langle \Psi_1^{(\eta_1 L)} \rangle_{\tau_1 L}}{L^1} \cdot \frac{P_{(\tau_2 - \tau_1)L} \Psi_1^{(\eta_2 L)} - \langle P_{(\tau_2 - \tau_1)L} \Psi_1^{(\eta_2 L)} \rangle_{\tau_1 L}}{L^1} \right) \right\rangle_{\tau_1 L} \\ \rightarrow \tau_1 \min(\eta_1, \eta_2).$$

This can be checked using Proposition 4.1. Assume without loss of generality that $\eta := \eta_1 \leq \eta_2$ and set $\tau = \tau_1$. Since $\langle E_{ii} E_{jj} \rangle = \langle E_{ii} \rangle \langle E_{jj} \rangle$ for $i \neq j$, then

$$\begin{aligned} & \lim_{L \rightarrow \infty} L^{-2} \left\langle \left(\sum_{i=1}^{\lfloor \eta_1 L \rfloor} (E_{ii} - \tau_1 L) \right) \left(\sum_{j=1}^{\lfloor \eta_2 L \rfloor} (E_{jj} - \tau_1 L) \right) \right\rangle_{\tau_1 L} \\ &= \lim_{L \rightarrow \infty} L^{-2} \left\langle \left(\sum_{i=1}^{\lfloor \eta_1 L \rfloor} E_{ii} - \eta_1 \tau_1 L^2 \right) \left(\sum_{j=1}^{\lfloor \eta_1 L \rfloor} E_{jj} - \eta_1 \tau_1 L^2 \right) \right\rangle_{\tau_1 L} \\ &= \lim_{L \rightarrow \infty} \frac{(\eta L(\tau^2 L^2 + \tau L) + \eta L(\eta L - 1)(\tau L)^2 - 2\eta\tau \cdot \eta\tau L^4 + \eta^2 \tau^2 L^4)}{L^2} \\ &= \eta\tau \end{aligned}$$

The remainder of this section will prove Theorem 5.2. By Theorem 5.1 of [6] and the fact that P_t preserves the centre, it is immediate that convergence to a Gaussian vector holds. It only remains to compute the covariance.

From the presence of Ψ_1^2 in $P_t \Psi_4$, it is necessary to understand products of Ψ_k . Heuristically, if $\Psi_1 \approx cL^2 + \xi L$, where ξ is a Gaussian random variable, then $\Psi_1^2 \approx c^2 L^4 + 2c\xi L^3$. Here are two examples which demonstrate this:

Example 9 Since

$$\lim_{L \rightarrow \infty} L^{-2} \langle \Psi_1^{(\eta L)} \rangle_{\tau L} = (\tau\eta - \frac{1}{2}\eta^2),$$

the heuristics would predict that

$$\begin{aligned} & \lim_{L \rightarrow \infty} \left\langle \frac{(\Psi_1^{(\eta L)} - \langle \Psi_1^{(\eta L)} \rangle_{\tau L}) \left([\Psi_1^{(\eta L)}]^2 - \langle [\Psi_1^{(\eta L)}]^2 \rangle_{\tau L} \right)}{L^4} \right\rangle_{\tau L} \\ &= (2\tau\eta - \eta^2) \lim_{L \rightarrow \infty} \left\langle \frac{(\Psi_1^{(\eta L)} - \langle \Psi_1^{(\eta L)} \rangle_{\tau L}) (\Psi_1^{(\eta L)} - \langle \Psi_1^{(\eta L)} \rangle_{\tau L})}{L^2} \right\rangle_{\tau L} \\ &= (2\tau\eta - \eta^2) \eta\tau \end{aligned}$$

And indeed, an explicit calculation yields

$$\begin{aligned}
& \lim_{L \rightarrow \infty} L^{-4} \left\langle \left(\sum_{i=1}^{\eta L} E_{ii} - \eta \tau L^2 \right) \sum_{j,k=1}^{\eta L} E_{jj} E_{kk} + \left(\sum_{i=1}^{\eta L} E_{ii} - \eta \tau L^2 \right) (-\eta^2 L^2) \sum_{j=1}^{\eta L} E_{jj} \right\rangle_{\tau L} \\
&= \lim_{L \rightarrow \infty} L^{-4} \left((\eta \tau L^2 + 3L^4 \eta^2 \tau^2 + \eta^3 \tau^3 L^6) - \eta \tau L^2 (\eta^2 \tau^2 L^4 + \eta \tau L^2) \right. \\
&\quad \left. - (\eta^2 L^2) (\eta^2 \tau^2 L^4 + \eta \tau L^2 - \eta \tau L^2 \cdot \eta \tau L^2) \right) \\
&= (2\tau\eta - \eta^2)\eta\tau.
\end{aligned}$$

Example 10 Consider Theorem 5.1 with $r = 2, k_1 = 3, k_2 = 4$. Using the formula for $P_t \Psi_4$ from Example 7 and replacing Ψ_1^2 with $(2\tau_1\eta_2 - \eta_2^2)\Psi_1$ yields

$$\begin{aligned}
& 12\eta_2\tau_1^2\tau_2(\eta_2^2\tau_1 + \tau_1(3\eta_2\tau_2 + 2\eta_2^2) + \tau_2(\tau_2 + 3\eta_2)(\tau_1 + \eta_1)) \\
&= 12\eta_2\tau_1^3(\eta_2^2\tau_1 + \tau_1(3\eta_2\tau_1 + 2\eta_2^2) + \tau_1(\tau_1 + 3\eta_2)(\tau_1 + \eta_1)) \\
&+ 4(\tau_2 - \tau_1) \cdot 3\eta_2\tau_1^2(\eta_2^2\tau_1 + 6\eta_2\tau_1^2 + 3\tau_1(\eta_2 + \tau_1)(\eta_1 + \tau_1)) \\
&+ (6(\tau_2 - \tau_1)^2 + 4(\tau_2 - \tau_1)\eta_2) \cdot 6\eta_2\tau_1^2(\tau_1(\tau_1 + \eta_1) + \eta_2\tau_1) \\
&+ (4(\tau_2 - \tau_1)^3 + 12(\tau_2 - \tau_1)^2\eta_2 + 2(\tau_2 - \tau_1)\eta_2^2) \cdot 3\eta_2\tau_1^2(\tau_1 + \eta_1) \\
&+ 2(\tau_2 - \tau_1) \cdot (2\tau_1\eta_2 - \eta_2^2) \cdot 3\eta_2\tau_1^2(\tau_1 + \eta_1),
\end{aligned}$$

which can be checked computationally.

Given a partition $\rho = (\rho_1, \dots, \rho_l)$, let its *weight* $\text{wt}(\rho)$ denote $|\rho| + l(\rho) = \rho_1 + \dots + \rho_l + l$, and let $\Psi_\rho = \prod_{i=1}^l \Psi_{\rho_i}$. In the asymptotic limit, we should be able to replace Ψ_ρ with a linear combination of Ψ_{ρ_i} . In the examples above, Ψ_1^2 was replaced with $(2\tau\eta - \eta^2)\Psi_1$.

Proposition 5.3. *Let $\eta, \tau > 0$ be fixed. (1) Set $N = \lfloor \eta L \rfloor$ and $t = \tau L$. Then $\langle \Psi_{\rho, N} \rangle_t = \Theta(L^{\text{wt}(\rho)})$.*

(2) *There exist constants $c'_{k,\rho}(\tau, \eta)$ such that*

$$P_{\tau L} \Psi_{k, N} = \sum_{\rho} (c'_{k,\rho}(\tau, \eta) + o(1)) L^{k+1-\text{wt}(\rho)} \Psi_{\rho}.$$

where the sum is over ρ with weight $\text{wt}(\rho) \leq k + 1$.

(3) *For any $\tau_1 > \tau_0$, there exist constants $c_{kj}(\tau_1, \tau_0, \eta)$ such that*

$$\begin{aligned}
& \lim_{L \rightarrow \infty} \left\langle \frac{\Psi_m - \langle \Psi_m \rangle_{\tau_0 L}}{L^m} \cdot \frac{P_{(\tau_1 - \tau_0)L} \Psi_k - \langle P_{(\tau_1 - \tau_0)L} \Psi_k \rangle_{(\tau_1 - \tau_0)L}}{L^k} \right\rangle_{\tau_0 L} \\
&= \lim_{L \rightarrow \infty} \sum_{j=1}^k c_{kj}(\tau_1, \tau_0, \eta) \left\langle \frac{\Psi_m - \langle \Psi_m \rangle_{\tau_0 L}}{L^m} \cdot \frac{\Psi_j - \langle \Psi_j \rangle_{\tau_0 L}}{L^j} \right\rangle_{\tau_0 L}
\end{aligned}$$

Proof. (1) This can be proved from [6], but this will be an alternative proof.

By definition,

$$\Psi_{\rho} = \sum_{m_1=1}^{\rho_1} \cdots \sum_{m_l=1}^{\rho_l} \sum_{\pi_1 \in \Pi_{\rho_1}^{(m_1)}} \cdots \sum_{\pi_l \in \Pi_{\rho_l}^{(m_l)}} E(\pi_1) \cdots E(\pi_l).$$

Consider the sum over l -tuples (π_1, \dots, π_l) such that the paths π_1, \dots, π_l cross over a total of exactly ν distinct vertices. There are $\binom{N}{\nu} = \Theta(L^\nu)$ such l -tuples, so it remains to estimate $\langle E(\pi_1) \cdots E(\pi_l) \rangle_{\tau L}$. Let π be the union of the paths π_1, \dots, π_l . Decompose π into the union of s simple cycles. By Proposition 4.1, $\langle E(\pi_1) \cdots E(\pi_l) \rangle_{\tau L} = \mathcal{O}(L^s)$. Decomposing π_j into s_j simple cycles, it is clear that $s = s_1 + \dots + s_l$. If π_j covers exactly ν_j vertices, then elementary graph theory gives $s_j = \rho_j - \nu_j + 1$. Since $\nu_1 + \dots + \nu_l \geq \nu$, thus

$$\langle \Psi_{\rho, N} \rangle_t = \mathcal{O}(L^\nu L^{s_1 + \dots + s_l}) = \mathcal{O}(L^{\rho_1 + \dots + \rho_l + l}).$$

To get a lower bound, just observe that the constant term in $\Psi_{\rho, N}$ is $\Theta(L^{\text{wt}(\rho)})$.

(2) By Theorem 3.1(5), $P_{\tau L} \Psi_k$ can be expressed as a linear combination of Ψ_ρ . Taking $\langle \cdot \rangle_L$ and using that $\langle P_{\tau L} X \rangle_L = \langle X \rangle_{(1+\tau)L}$, it follows from (1) that only $\text{wt}(\rho) \leq k+1$ terms have nonzero coefficients.

(3) First apply (2) to the left-hand-side. Then, by (1),

$$\begin{aligned} & \Psi_{\rho_1} \cdots \Psi_{\rho_l} - \langle \Psi_{\rho_1} \cdots \Psi_{\rho_l} \rangle_{\tau_0 L} \\ &= \sum_{j=1}^l \langle \Psi_{\rho_1} \rangle_{\tau_0 L} \cdots \langle \widehat{\Psi_{\rho_j}} \rangle_{\tau_0 L} \cdots \langle \Psi_{\rho_l} \rangle_{\tau_0 L} \left(\Psi_{\rho_j} - \langle \Psi_{\rho_j} \rangle_{\tau_0 L} \right) + \text{smaller order terms} \end{aligned}$$

□

Given a Laurent polynomial $p(w)$, let $p(w)[w^r]$ denote the coefficient of w^r in $p(w)$. Using the expansion $(z-w)^{-2} = z^{-2}(1 + 2(w/z) + 3(w/z)^2 + \dots)$ in Theorem 5.1 and taking residues, one obtains

$$\sum_{l=1}^k c_{kl}(\tau_2, \tau_1, \eta_2)(\eta_2 w^{-1} + \tau_1 + \tau_1 w)^l [w^r] = (\eta_2 w^{-1} + \tau_2 + \tau_2 w)^k [w^r], r \leq -1.$$

For example, for $k=3$ and $r=-1$, and using the expansion of $P_t \Psi_3$, this says

$$1 \cdot (3\eta_2^2 \tau_1 + 3\eta_2 \tau_1^2) + 3(\tau_2 - \tau_1) \cdot 2\eta_2 \tau_1 + 3((\tau_2 - \tau_1)^2 + (\tau_2 - \tau_1)\eta_2) \cdot \eta_2 = 3\eta_2^2 \tau_2 + 3\eta_2 \tau_2^2. \quad (9)$$

We need a formula for $r \geq 1$. Theorem 5.2 follows from the proposition below.

Proposition 5.4. *For $r \geq 1$,*

$$\sum_{l=1}^k c_{kl}(\tau_2, \tau_1, \eta_1)(\eta_1 z^{-1} + \tau_1 + \tau_1 z)^l [z^r] = (\eta_1 \frac{\tau_2}{\tau_1} z^{-1} + \tau_2 + \tau_1 z)^k [z^r].$$

Proof. We start with an illustrative example. For $k=3$ and $r=1$ we would want to show

$$1 \cdot (3\eta_1 \tau_1^2 + 3\tau_1^3) + 3(\tau_2 - \tau_1) \cdot 2\tau_1^2 + 3((\tau_2 - \tau_1)^2 + (\tau_2 - \tau_1)\eta_1) \cdot \tau_1 = 3\eta_1 \tau_1 \tau_2 + 3\tau_1 \tau_2^2. \quad (10)$$

This can be checked directly, but in general the coefficients c_{kl} are difficult to work with. Instead, we would like to show that it follows directly from

the covariance formula along space-like paths. Indeed, this can be done just by multiplying (9) by $(\tau_1/\eta_1)^r$. (And recall that $\eta_2 < \eta_1$ in (9) while $\eta_1 < \eta_2$ in (10)).

Let $S_l^{(r)} = \{(\epsilon_1, \dots, \epsilon_l) \in \{-1, 0, +1\}^l : \epsilon_1 + \dots + \epsilon_l = r\}$ and define

$$\chi(j) = \begin{cases} \eta_1, j = -1 \\ \tau_1, j = 0 \\ \tau_1, j = 1 \end{cases} \quad \chi'_{\text{ti}}(j) = \begin{cases} \eta_1 \frac{\tau_2}{\tau_1}, j = -1 \\ \tau_2, j = 0 \\ \tau_1, j = 1 \end{cases} \quad \chi'_{\text{sp}}(j) = \begin{cases} \eta_1, j = -1 \\ \tau_2, j = 0 \\ \tau_2, j = 1 \end{cases}$$

With this notation, what we want to show is that

$$\sum_{l=1}^k c_{kl}(\tau_2, \tau_1, \eta_1) \sum_{\vec{\epsilon} \in S_l^{(r)}} \prod_{j=1}^l \chi(\epsilon_j) = \sum_{\vec{\epsilon}' \in S_k^{(r)}} \prod_{j=1}^k \chi'_{\text{ti}}(\epsilon'_j), r \geq 1. \quad (11)$$

From (10),

$$\sum_{l=1}^k c_{kl}(\tau_2, \tau_1, \eta_1) \sum_{\vec{\epsilon} \in S_l^{(-r)}} \prod_{j=1}^l \chi(\epsilon_j) = \sum_{\vec{\epsilon}' \in S_k^{(-r)}} \prod_{j=1}^k \chi'_{\text{sp}}(\epsilon'_j), r \geq 1.$$

By sending $\epsilon_j \mapsto -\epsilon_j$, this is equivalent to

$$\sum_{l=1}^k c_{kl}(\tau_2, \tau_1, \eta_1) \sum_{\vec{\epsilon} \in S_l^{(r)}} \prod_{j=1}^l \chi(-\epsilon_j) = \sum_{\vec{\epsilon}' \in S_k^{(r)}} \prod_{j=1}^k \chi'_{\text{sp}}(-\epsilon'_j), r \geq 1.$$

And since for all r ,

$$\sum_{l=1}^k c_{kl}(\tau_2, \tau_1, \eta_1) \sum_{\vec{\epsilon} \in S_l^{(r)}} \prod_{j=1}^l \chi(-\epsilon_j) = \left(\frac{\eta_1}{\tau_1}\right)^r \sum_{l=1}^k c_{kl}(\tau_2, \tau_1, \eta_1) \sum_{\vec{\epsilon} \in S_l^{(r)}} \prod_{j=1}^l \chi(\epsilon_j)$$

it thus follows that the left-hand-side of (11) equals

$$\left(\frac{\tau_1}{\eta_1}\right)^r \sum_{\vec{\epsilon}' \in S_k^{(r)}} \prod_{j=1}^k \chi'_{\text{sp}}(-\epsilon'_j), r \geq 1.$$

So it suffices to show that

$$\left(\frac{\tau_1}{\eta_1}\right)^r \prod_{j=1}^k \chi'_{\text{sp}}(-\epsilon'_j) = \prod_{j=1}^k \chi'_{\text{ti}}(\epsilon'_j), \quad \text{for all } \vec{\epsilon}' \in S_k^{(r)}, r \geq 1.$$

Since $r = |\{\epsilon_j = 1\}| - |\{\epsilon_j = -1\}|$, it follows that the left-hand-side is

$$\left(\frac{\tau_1}{\eta_1}\right)^r \cdot \eta_1^{|\epsilon_j=1|} \tau_2^{|\epsilon_j=0|} \tau_2^{|\epsilon_j=-1|} = \tau_1^r \eta_1^{|\epsilon_j=-1|} \tau_2^{|\epsilon_j=0|} \tau_2^{|\epsilon_j=-1|}.$$

And similarly, the right-hand-side is

$$\tau_1^{|\epsilon_j=1|} \tau_2^{|\epsilon_j=0|} \left(\eta_1 \frac{\tau_2}{\tau_1}\right)^{|\epsilon_j=-1|} = \tau_1^r \eta_1^{|\epsilon_j=-1|} \tau_2^{|\epsilon_j=0|} \tau_2^{|\epsilon_j=-1|}.$$

□

The formula in Theorem 5.2 appears to be different from the formula in [5]. In particular, the covariance along space-like paths is different from the covariance along time-like paths. However, after rescaling from Brownian Motion to Ornstein–Uhlenbeck, i.e. replacing τ_i, τ_j with $e^{2\tau_i}, e^{2\tau_j}$ and multiplying by $e^{-\tau_j k_j} e^{-\tau_i k_i}$, the formula becomes

$$\mathbb{E}[\xi_i \xi_j] = \begin{cases} -\frac{1}{\pi} \frac{e^{\tau_j}}{e^{\tau_i}} \int \int_{|z| > |w|} (\eta_i z^{-1} + e^{\tau_i} + z)^{k_i} (\eta_j w^{-1} + e^{\tau_j} + w)^{k_j} \left(\frac{e^{\tau_j}}{e^{\tau_i}} z - w\right)^{-2} dz dw, \\ \quad \eta_i \geq \eta_j, \tau_i \leq \tau_j \\ -\frac{1}{\pi} \frac{e^{\tau_j}}{e^{\tau_i}} \int \int_{|z| > |w|} (\eta_j z^{-1} + e^{\tau_j} + z)^{k_j} (\eta_i w^{-1} + e^{\tau_i} + w)^{k_i} \left(\frac{e^{\tau_j}}{e^{\tau_i}} z - w\right)^{-2} dz dw, \\ \quad \eta_i < \eta_j, \tau_i \leq \tau_j \end{cases}$$

In both expressions, the z -contour is larger and corresponds to the higher level (η_i in the first case and η_j in the second). Hence, by switching the subscripts i and j in η , the formula is the same in both cases. It also matches the formula in [5] with the expression $e^{\tau_j - \tau_i}$ playing the role of $c(t_p, t_q)$.

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